SECTION 3.6: THE DERIVATIVE AS A RATE OF CHANGE

EXAMPLE 1: Suppose $T(t)$ represents the temperature (in degrees Fahrenheit) t hours after 10 AM. Suppose it is 45°F at noon and 48°F at 2 PM.

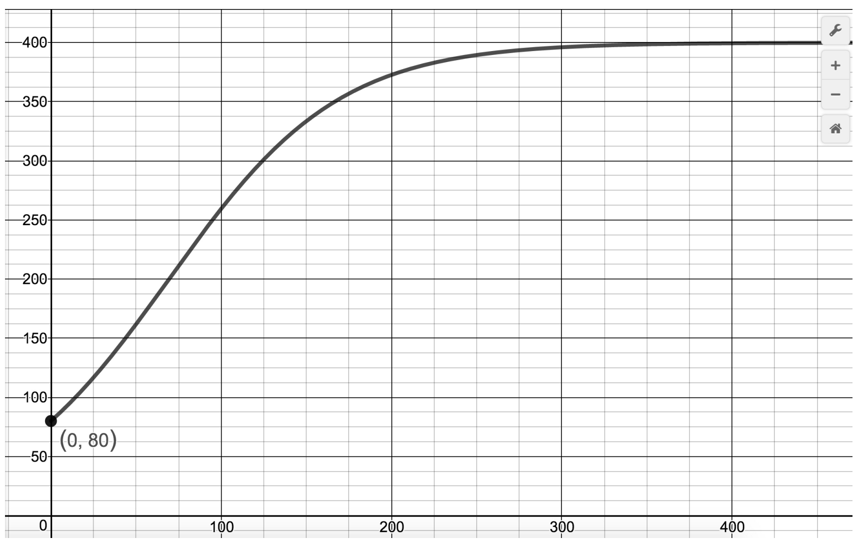
- Approximate $T'(4)$.

Ans: $T'(4) \approx 1.5$

- Use $T'(4)$ to approximate the temperature at 5 PM.

Ans: 52.5°F

EXAMPLE 2: The graph below models the population of black bears in a certain county, $P(t)$, t years after they were reintroduced in the year 2000. Use the graph to answer the following questions.



$y = P(t)$

- Approximate $\lim_{t\to\infty} P(t)$. What does your answer mean in terms of the black bear population?

$\lim_{t\to\infty} P(t) \approx 400$; the black bear population will level off at around 400 bears.

- Which is greater, $P'(100)$ or $P'(300)$? What does this mean in terms of the black bear population?

$P'(100) > P'(300)$ which means the population of bears is growing faster in 2100 than in 2300.

- Is $P''(200)$ positive or negative? What does this mean in terms of the black bear population?

$P''(200) < 0$ which means the population growth rate of bears is decreasing.

ONE DIMENSIONAL MOTION

RECALL: If $s(t)$ represents the position of an object at time t, then:

- $s'(t) = v(t)$ is the velocity at time t: this is how fast and in what direction the object is traveling.
- $|v(t)|$ is the speed of the object at time t: this is how fast the object is traveling.
- $s''(t) = v'(t) = a(t)$ is the acceleration at time t: this tells us the rate of change of the object's velocity.

  NOTE: If $v(t)$ and $a(t)$ have the **same sign**, the object is **speeding up**.

  If $v(t)$ and $a(t)$ have **opposite signs**, the object is **slowing down**.

EXAMPLE 3: Suppose an object travels along the x-axis according to the formula: $s(t) = 6t^2 - t^3$, $t \geq 0$.

- Find expressions for the velocity function, $v(t)$, and acceleration function, $a(t)$.

  For velocity: $v(t) = s'(t) = D_t\left[6t^2 - t^3\right] = 12t - 3t^2$.

  For acceleration: $a(t) = s''(t) = v'(t) = D_t\left[12t - 3t^2\right] = 12 - 6t$.

- When $t = 5$: where is the object? which direction is it moving? is the object speeding up or slowing down?

  To find where the object is when $t = 5$, we find $s(5) = 6(5)^2 - (5)^3 = 25$. So the object is at $x = 25$.

  To find which direction the object is moving, we find $v(5) = 12(5) - 3(5)^2 = -15$.

  Since $v(5)$ is negative, the object is moving in the direction of **decreasing** x-values, that is, to the **left**.

  To determine if the object is speeding up or slowing down, we compute $a(5) = 12 - 6(5) = -18$.

  Since $a(5)$ is **negative**, the velocity is **decreasing**.

  Since $v(5)$ is also **negative**, the velocity is becoming 'more' negative, so the object is **speeding up**.

- Solve and interpret $v(t) = 0$.

  Solving $v(t) = 12t - 3t^2 = 0$ gives $3t(4 - t) = 0$ so $t = 0$ or $t = 4$. These are the times the object is 'at rest.'

- Make a sign diagram for $v(t)$ and use this to help describe the journey of the object.

  We know $v(t) = 0$ when $t = 0$ and $t = 4$ and so we test the sign of $v(t)$ in the intervals $(0, 4)$ and $(4, \infty)$. We find $v(1) = 12(1) - 3(1)^2 = 9$ and we've already found $v(5) = -15$. Hence, our sign diagram is:

$$\begin{array}{ccccc} 0 & (+) & 0 & (-) & v(t) \\ \hline 0 & & 4 & & t \end{array}$$

  $v(t)$ is $(+)$ on the interval $(0, 4)$ which means the object is moving the the direction of **increasing** x-values. In other words, the object is moving to the **right** on the interval $(0, 4)$.

  Likewise, $v(t)$ is $(-)$ on the interval $(0, \infty)$ which means the object is moving the the direction of **decreasing** x-values. In other words, the object is moving to the **left** on the interval $(4, \infty)$.

  Putting this together we have the object starts at $x = s(0) = 6(0)^2 - (0)^3 = 0$, travels to the right to $x = s(4) = 6(4)^2 - (4)^3 = 32$, turns around, then heads back to the left forever.

EXAMPLE 4: (VIDEO) Below is the graph of the height of a paper airplane off of the ground, $s(t)$, measured in feet, t seconds after it is launched into the air.

$y = s(t)$

- Approximate the average velocity for the first 7 seconds of flight: $0 \leq t \leq 7$.

  Is your answer a good approximation of $v(t)$ over this time interval? Explain.

  Ans: average velocity: $\frac{3}{7}$ feet per second.

- What is the highest altitude the paper airplane reached? When did that happen?

  Ans: highest altitude: 11.25 feet 15 seconds after launch.

- How long was the paper airplane in flight?

  Ans: The paper airplane was in flight for 20 seconds.

- When is $v(t) > 0$? When is $v(t) < 0$? When is $v(t) = 0$? What about $|v(t)|$?

  Ans: $v(t) > 0$ for $0 < t < 15$; $v(t) < 0$ for $15 < t \leq 20$; $v(t) = 0$ when $t = 0, 15$;

  $|v(t)| > 0$ for $0 < t < 15$ and $15 < t \leq 20$.

- Is $a(t)$ ever positive? Explain.

  Ans: $a(t) > 0$ for $0 \leq t < 10$

EXAMPLE 5:(VIDEO) An object travels along the y-axis according to: $s(t) = 2t^3 - 21t^2 + 60t$, for $t \geq 0$.

- Find expressions for the velocity function, $v(t)$, and the acceleration function, $a(t)$.

Ans: $v(t) = 6t^2 - 42t + 60$; $a(t) = 12t - 42$

- When $t = 1$: where is the object? which direction is it moving? is the object speeding up or slowing down?

Ans: When $t = 1$, the object is at $y = 41$, moving upwards and slowing down.

- Determine when the object is 'at rest.'

Ans: The object is at rest when $t = 2$ or $t = 5$.

- Make a sign diagram for $v(t)$ and use this to help describe the journey of the object.

$$\begin{array}{ccccccc} & (+) & 0 & (-) & 0 & (+) & v(t) \\ \hline 0 & & 2 & & 5 & \rightarrow & t \end{array}$$

The object starts off at $y = s(0) = 0$ and starts heading upwards to $y = s(2) = 52$.

The object then turns around and heads downwards until it reaches $y = s(5) = 25$.

At this point, the object turns around one last time and heads upwards forever more.

BUSINESS APPLICATIONS: REVIEW OF BASIC CONCEPTS

RECALL:

- The **cost** to produce x items is denoted $C(x)$.
- The **average cost** to produce x items is denoted $\overline{C}(x) = \dfrac{C(x)}{x}$.
- The **price-demand** function, denoted $p(x)$, gives the price per item to charge in order to sell x items.
- The **revenue** function, denoted $R(x)$, is the money collected by selling x items. That is, $R(x) = x\,p(x)$.
- The **profit** function, denoted $P(x)$ is the money earned after cost by producing and selling x items. That is, $P(x) = R(x) - C(x)$.

EXAMPLE 6: Consider the economic scenario described below:

The cost in dollars, $C(x)$, to make x "I'd rather be a Sasquatch" T-shirts is: $C(x) = 2x + 26$, $x \geq 0$.

The price charged per T-Shirt (in dollars per shirt) is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.

- Find and interpret $C(6)$, $\overline{C}(6)$, and $p(6)$.

  $C(6) = 2(6) + 26 = 38$. It costs \$38 to make 6 shirts.

  $\overline{C}(6) = \dfrac{C(6)}{6} = 6.\overline{3}$. It costs an average of $\approx$ \$6.33 per shirt to make the first 6 shirts.

  $p(6) = 30 - 2(6) = 18$. In order to sell 6 shirts, the price should be set at \$18 per shirt.

- Find an expression for the revenue function, $R(x)$ interpret $R(6)$.

  $R(x) = x\,p(x) = x(30 - 2x) = -2x^2 + 30x$. Since $p(x)$ is restricted to $0 \leq x \leq 15$, so is $R(x)$.

  $R(6) = -2(6)^2 + 30(6) = 108$. If 6 shirts are sold, the revenue is \$108.

  NOTE: We know $p(6) = 18$, so we check: $R(6) = 6\,p(6) = 6 \cdot 18 = 108$ ✓

- Find an expression for the profit function, $P(x)$ and interpret $P(6)$.

  $P(x) = R(x) - C(x) = \left(-2x^2 + 30x\right) - (2x + 26) = -2x^2 + 28x - 26$.

  Since $C(x)$ is valid for $x \geq 0$ and $R(x)$ is valid for $0 \leq x \leq 15$, $P(x)$ is valid for $0 \leq x \leq 15$.

  $P(6) = -2(6)^2 + 28(6) - 26 = 70$ which means if 6 shirts are produced and sold, the profit is \$70.

  NOTE: We know $R(6) = 108$ and $C(6) = 38$, so we check: $P(6) = R(6) - C(6) = 108 - 38 = 70$ ✓

- How many shirts should be sold to maximize profit? What is the maximum profit?
  How much should be charged per shirt to achieve the maximum profit?

  We know from algebra that the graph of $P(x) = -2x^2 + 28x - 26$ is a downwards facing parabola.

  Hence, the maximum profit corresponds to the vertex of the parabola.

  Using the vertex formula, we find $x = -\dfrac{b}{2a} = -\dfrac{28}{2(-2)} = 7$. So selling 7 shirts maximizes the profit.

  We find $P(7) = -2(7)^2 + 28(7) - 26 = 72$, so the maximum profit obtainable is \$72.

  We find $p(7) = 30 - 2(7) = 16$, so to sell 7 shirts, we need to charge \$16 per shirt.

BUSINESS APPLICATIONS: MARGINALS

We know that for 'small' values of h,

$$C'(x) \approx \frac{C(x+h) - C(x)}{h}, \quad R'(x) \approx \frac{R(x+h) - R(x)}{h}, \quad \text{and} \quad P'(x) \approx \frac{P(x+h) - P(x)}{h}$$

Taking $h = 1$, we get:

$$C'(x) \approx C(x+1) - C(x), \quad R'(x) \approx R(x+1) - R(x), \quad \text{and} \quad P'(x) \approx P(x+1) - P(x)$$

In other words, for a given number of items being produced, x, the value of $C'(x)$ is approximately the change in cost associated with producing **one additional item**. Similarly, if x items are sold, the value of $R'(x)$ approximates the change in revenue obtained by selling one additional item and the value of $P'(x)$ approximates the change in profit obtained by producing and selling one additional item.

DEFINITIONS:

- The **marginal cost** is $MC(x) = C'(x) \cdot 1\,\text{item}$.

  $MC(x)$ approximates the change in cost associated with producing one additional item.

- The **marginal revenue** is $MR(x) = R'(x) \cdot 1\,\text{item}$.

  $MR(x)$ approximates the change in revenue obtained by selling one additional item.

- The **marginal profit** is $MP(x) = P'(x) \cdot 1\,\text{item}$.

  $MP(x)$ approximates the change in profit obtained by producing and selling one additional item.

  NOTE: Since $P(x) = R(x) - C(x)$, $P'(x) = R'(x) - C'(x)$, so $MP(x) - MR(x) - MC(x)$.

EXAMPLE 7: Answer the following questions by referring to the economic scenario detailed above.

- Find $MC(7)$ and compare this to $\overline{C}(7)$.

  $MC(x) = C'(x) = D_x[2x + 26] = 2$. This means the cost to make the 8th shirt is approximately \$2 .

  $\overline{C}(7) = \dfrac{C(7)}{7} = \dfrac{40}{7} \approx 5.71$ which means that when making 7 shirts, the average cost per shirt is $\approx$ \$5.17.

  Since the marginal cost is less than the average cost, this means the average cost should continue to drop if another shirt is made.

- Find and interpret $MR(7)$.

  $MR(x) = D_x\left[-2x^2 + 30x\right] = -4x + 30$. Hence, $MR(7) = -4(7) + 30 = 2$.

  This means if 7 shirts are sold, the additional revenue obtained by selling the 8th shirt is approximately \$2.

  NOTE: The actual change in revenue is: $R(8) - R(7) = 112 - 112 = 0$.

  In other words, the revenue is the same if selling 7 shirts or 8 shirts. How can this be?

- Find and interpret $MP(7)$.

  $MP(x) = D_x\left[-2x^2 + 28x - 26\right] = -4x + 28$. Hence, $MP(7) = -4(7) + 28 = 0$.

  This means if 7 shirts are sold, there is no additional profit obtained by making and selling the 8th shirt (!)

  Since $MR(7) = 2 = MC(7)$, what's happening here is the additional revenue obtained by **selling** the 8th shirt is phased out by the additional cost incurred by **producing** the 8th shirt.

  NOTE 1: The actual change in profit is: $P(8) - P(7) = 70 - 72 = -2$.

  This means we'd actually **lose** \$2 in profit if we made and sold the 8th shirt.

  NOTE 2: It is not a coincidence that $MP(7) = 0$ and the maximum profit is obtained when $x = 7$.

  We'll revisit this phenomenon later. Stay tuned!

HOMEWORK: Section 3.6: 1 - 31 odd, 35, 41, 57*, 59*